Math 10A with Professor Stankova
Worksheet, Discussion #19; Monday, 10/9/2017
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Fundamental Theorem of Calculus II

1. True **FALSE** $\int_a^x f(u)du$ gives you a general form of an antiderivative (including the $+C$).
2. **TRUE** False Let $F(x) = \int_0^x f(u)du$. Then $G(x)$ be another antiderivative of $f(x)$. For all x we have $F(x) = G(x) - G(0)$.
3. **TRUE** False Let $f(x)$ be a continuous function on the interval $[a, b]$, and let $F(x) = \int_a^x f(u)du$. Then $F(x)$ is defined on the interval $[a, b]$.
4. True **FALSE** Let $f(x)$ be a continuous function on the interval $[a, b]$, and let $F(x) = \int_a^x f(u)du$. Then $F'(x) = f(x)$ on the interval $[a, b]$.
5. If $\int_1^x f(u)du = \frac{1}{x} + a$, find f, a.

Solution: Taking the derivative, the left side gives us $f(x)$ and the right side gives us $-x^{-2}$ so $f(x) = -x^{-2}$. Then we have that $\int_1^x f(u)du = 1/x - 1/1$ so $a = 1$.

$\Sigma-$notation

Examples

6. Write out $\sum_{r=1}^{n}(-1)^r r$.

Solution: The first few terms are $(-1)^1 \cdot 1 + (-1)^2 \cdot 2$ so we have that this is equal to

$$-1 + 2 - 3 + \cdots + (-1)^n n.$$

7. Write out $\sum_{r=1}^{\infty}(-1)^r r$.

Solution:

$$-1+2-3+\cdots.$$

8. Write $1+3+5+\cdots+(2n+1)$ in $\sum$ notation.

Solution: We choose an index i and let it start at 1. We notice that this is an arithmetic sequence with difference 2 and so we guess that each term is something like $2i+c$. If $i=1$, then we should get 1 so $c=-1$ and if $i=2$, then $2i-1=3$ as required. We want to end when $2i-1=2n+1$ so when $i=n+1$, thus

$$1+3+\cdots+(2n+1)=\sum_{i=1}^{n+1}(2i-1).$$

Problems

9. Write out $\displaystyle\sum_{k=1}^{2n}\frac{1}{k}$.

Solution:

$$\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{2n}.$$

10. Write out $\displaystyle\sum_{a=1}^{n}f(a)^2$.

Solution:

$$f(1)^2+f(2)^2+\cdots+f(n)^2.$$

11. Convert $\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\cdots$ into Σ notation.

Solution: We see that the pattern is similar to $\frac{(-1)^i}{i}$, but when $i=1$, the term is positive so we need to multiply by an additional factor of -1. Since this term doesn't end, this is

$$\sum_{i=1}^{\infty}-\frac{(-1)^i}{i}.$$

12. Convert $(f(x)-1)^2+(f(x)-2)^2+\cdots+(f(x)-10)^2$ to Σ notation.

Solution: Each term is of the form $(f(x)-i)^2$ with i starting at 1 and going to 10 so this is

$$\sum_{i=1}^{10}(f(x)-i)^2.$$

13. Convert $-1+4-9+\cdots-121$ into Σ notation.

Solution: We recognize these terms as squares. Then note that the signs alternate so there is a factor of $(-1)^t$ in there as well. Verifying that $-121=(-1)^{11}\cdot 11^2$, this sum is

$$\sum_{t=1}^{11}(-1)^t t^2.$$

14. Write $1+2+4+8+\cdots+2^{2^n}$ in Σ notation.

Solution: Each term is of the form 2^k where we start at $k=0$ and end at $k=2^n$ so the sum is

$$\sum_{k=0}^{2^n}2^k.$$

Substitution Rule

Example

15. Find $\displaystyle\int xe^{x^2}dx$.

Solution: Let $u=x^2$, then $du=2xdx$ and hence $xdx=\frac{du}{2}$. Therefore

$$\int xe^{x^2}dx=\int\frac{e^u du}{2}=\frac{e^u}{2}+C=\frac{e^{x^2}}{2}+C.$$

16. Find $\int_0^{16} \sqrt{4-\sqrt{x}}dx$.

Solution: We guess that $u = 4 - \sqrt{x}$ but $du = -\frac{1}{2\sqrt{x}}dx$ and it seems like we are stuck since that doesn't appear. But, remember that $u = 4 - \sqrt{x}$ so $\sqrt{x} = 4 - u$ and so $du = \frac{-1}{2(4-u)}dx$. When $x = 0$, then $u = 4$ and when $x = 16$, then $u = 4 - \sqrt{16} = 0$. Thus

$$\int_0^{16} \sqrt{4-\sqrt{x}}dx = \int_4^0 \sqrt{u}(2u-8)du = \frac{4}{5} \cdot u^{5/2} - \frac{16}{3} \cdot u^{3/2}\Big|_4^1$$
$$= 0 - (-128/5 - 128/3) = \frac{256}{15}.$$

Problems

17. Find $\displaystyle\int \frac{\ln x}{x}dx$.

Solution: Let $u = \ln x$, then $du = \frac{dx}{x}$, so we have that

$$\int \frac{\ln x}{x}dx = \int udu = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C.$$

18. Find $\displaystyle\int \frac{1}{x\ln x}dx$.

Solution: Let $u = \ln x$, then $du = \frac{1}{x}dx$ so

$$\int \frac{1}{x\ln x}dx = \int \frac{1}{u}du = \ln|u| + C = \ln|\ln x| + C.$$

19. Find $\displaystyle\int x\sqrt{1-x}dx$.

Solution: Let $u = 1 - x$ and so $du = -dx$ and $x = 1 - u$ so this is

$$\int x\sqrt{1-x} = \int (1-u)\sqrt{u}(-du) = \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C = \frac{2(1-x)^{5/2}}{5} - \frac{2(1-x)^{3/2}}{3} + C.$$

20. Find $\int_0^{\sqrt{\pi}} x\cos(x^2)dx$.

Solution: Let $u = x^2$ and so $du = 2xdx$ and when $x = 0$, then $u = 0$ and when $x = \sqrt{\pi}$, then $u = \pi$ so we have

$$\int_0^{\sqrt{\pi}} x\cos(x^2)dx = \int_0^{\pi} \frac{\cos(u)du}{2} = \frac{\sin u}{2}|_0^{\pi} = 0.$$

21. Find $\int \sin(x)\sec^2(x)dx$.

Solution: We rewrite $\sec^2(x) = \frac{1}{\cos^2(x)}$. Let $u = \cos(x)$ so that $du = -\sin x dx$ and hence

$$\int \sin(x)\sec^2(x)dx = \int -u^{-2}du = \frac{1}{u} + C = \frac{1}{\cos(x)} + C = \sec(x) + C.$$

22. Find $\int 2xe^{e^{x^2}}e^{x^2}dx$.

Solution: We first try $u = x^2$ so $du = 2xdx$ and hence

$$\int 2xe^{e^{x^2}}e^{x^2}dx = \int e^{e^u}e^u du.$$

Now let $v = e^u$ so $dv = e^u du$ and hence

$$= \int e^v dv = e^v + C = e^{e^u} + C = e^{e^{x^2}} + C.$$